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Complete minimal Möbius strips in $\mathbb{R}^n$ and the Björling problem

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Abstract

We use the solution to the Björling problem for minimal surfaces in $\mathbb{R}^n$ to classify the complete minimal Möbius strips in $\mathbb{R}^n$ whose total curvature is finite and of critical value for Gackstatter's inequality.

AMS subject classification: 53A10; 53C42

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1. Introduction

The classical Björling problem for minimal surfaces in $\mathbb{R}^3$ (see [3,11] for instance) goes back to 1844, and asks for the construction of a minimal surface in $\mathbb{R}^3$ containing a given curve in $\mathbb{R}^3$ with a given unit normal along it. The solution to this problem was obtained by Schwarz in 1890, by using holomorphic data. Some extensions of this classical Björling problem to other geometric theories, as well as some global applications of it have been developed in [1,2,5–7]. In particular, the extension of the Björling problem to $\mathbb{R}^n$ consists on the following:

> Let $\beta(s)$ be a regular analytic curve in $\mathbb{R}^n$, and let $\Pi(s)$ denote an analytic distribution of oriented planes along $\beta(s)$ such that $\beta'(s) \in \Pi(s)$ for all s. Find all minimal surfaces in $\mathbb{R}^n$ that contain $\beta(s)$ and such that its tangent plane distribution along $\beta(s)$ is given by $\Pi(s)$.

The objective of this work is to apply the Björling problem to the global study of the minimal surfaces in $\mathbb{R}^n$ with the topology of a Möbius strip. More concretely, we shall be interested in the minimal Möbius strips of $\mathbb{R}^n$ that are complete and of finite total curvature.

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We have organized the present paper as follows. In Section 2 we formulate in a global way the general solution to the Björling problem in $\mathbb{R}^n$, and use it to provide a general description free of the period problem for the minimal Möbius strips of $\mathbb{R}^n$. In Section 3 we use the Björling problem to study the complete minimal Möbius strips of finite total curvature in $\mathbb{R}^n$, showing that any such surface is the solution to a particular Björling problem in which the initial data are expressed as vector trigonometric polynomials. At last, in Section 4 we give a geometric construction of all complete minimal Möbius strips of finite total curvature in $\mathbb{R}^n$ that attain equality in a fundamental inequality of the theory: the Gackstatter inequality [4,12].

2. Möbius strips and the Björling problem

Definition 1. A pair of Björling data in $\mathbb{R}^n$ is a regular analytic curve $\beta : I \to \mathbb{R}^n$ together with an analytic vector field $B : I \to \mathbb{R}^n$ along β so that $\langle \beta', \beta' \rangle = \langle B, B \rangle > 0$ and $\langle \beta', B \rangle = 0$.

Obviously, any pair of initial data $\{\beta(s), \Pi(s)\}$ for the Björling problem describe a pair of Björling data given by $\beta(s)$ and $B(s) = J\beta'(s)$, being $J\beta'(s)$ the $\pi/2$-rotation of $\beta'(s)$ in the oriented tangent plane $\Pi(s)$.

Theorem 2 (Alías and Mira [2]). *Let $\beta(s)$, $B(s)$ be Björling data on a real interval I. There is a unique solution to the Björling problem in $\mathbb{R}^n$ for the data $\beta(s)$, $B(s)$.*

Such minimal surface can be constructed in a neighbourhood of $\beta(s)$ as

$$\psi(s + it) = \operatorname{Re} \beta(z) + \operatorname{Im} \int_{s_0}^{z} B(w)\, \mathrm{d}w. \tag{2.1}$$

Here $\beta(z)$, $B(z)$ are holomorphic extensions of $\beta(s)$, $B(s)$ over a simply connected domain $\Omega \subseteq \mathbb{C}$ containing I, and the integral is taken along an arbitrary path in Ω joining a fixed base point $s_0 \in I$ with $z = s + it$.

It is possible to give a global reformulation of the holomorphic representation given by the above theorem. Specifically, let $\psi : \Sigma \to \mathbb{R}^n$ be a minimal surface, let $\Gamma \subset \Sigma$ be a regular analytic curve and define $\beta = \psi(\Gamma)$. Then there is a real analytic one-form $\mathcal{B}$ along Γ which is tangent to the immersion, and such that $\langle \mathrm{d}\beta, \mathcal{B} \rangle = 0$ and $\langle \mathrm{d}\beta, \mathrm{d}\beta \rangle = \langle \mathcal{B}, \mathcal{B} \rangle$. This one-form may be chosen so that the Weierstrass one-form $\Phi = \psi_z\, \mathrm{d}z$ of ψ restricted to Γ verifies $\Phi|_\Gamma = \sigma$ for $\sigma = \mathrm{d}\beta - \mathrm{i}\mathcal{B}$. Hence, σ extends holomorphically to Σ and its extension Φ recovers the immersion ψ via the Weierstrass representation: $\psi = 2\operatorname{Re} \int \Phi : \Sigma \to \mathbb{R}^n$.

Conversely, let Σ be a Riemann surface and $\Gamma \subset \Sigma$ be a regular analytic curve. Define then *Björling data* $\beta, \mathcal{B}$ so that $\beta : \Gamma \to \mathbb{R}^n$ is a regular analytic curve and $\mathcal{B}$ is an analytic one-form along β verifying $\langle \mathrm{d}\beta, \mathcal{B} \rangle = 0$ and $\langle \mathrm{d}\beta, \mathrm{d}\beta \rangle = \langle \mathcal{B}, \mathcal{B} \rangle$. Let $\sigma = \mathrm{d}\beta - \mathrm{i}\mathcal{B}$, and assume:

1. σ admits a global extension Φ on Σ as a holomorphic one-form.
2. $||\Phi|| > 0$.
3. Φ has no real periods.

Then $\psi = \operatorname{Re} \int^z \Phi : \Sigma \to \mathbb{R}^n$ is a minimal surface such that $\psi|_\Gamma = \beta$, and its cotangent plane at every point of Γ is spanned by $\mathrm{d}\beta$ and $\mathcal{B}$.

To treat non-orientable minimal surfaces in $\mathbb{R}^n$ we first need to make some preliminary remarks. We refer to [12] for the details.

Let $\psi' : \mathcal{M} \to \mathbb{R}^n$ be a minimal immersion of a non-orientable surface $\mathcal{M}$ in $\mathbb{R}^n$. Then the two-sheeted oriented cover of $\mathcal{M}$, denoted by Σ, inherits naturally a Riemann surface structure and we have a canonical projection $p : \Sigma \to \mathcal{M}$. We can also define a map $I : \Sigma \to \Sigma$ that verifies $p \circ I = p$, and which is actually an antiholomorphic involution on Σ without fixed points. Then $\mathcal{M}$ can be identified naturally with the pair (Σ, I).

In this way, if $\psi : \Sigma \to \mathbb{R}^n$ is a minimal surface in $\mathbb{R}^n$ and there is an antiholomorphic involution $I : \Sigma \to \Sigma$ without fixed points such that $\psi \circ I = \psi$, then we can define $\psi' : \mathcal{M} \equiv (\Sigma, I) \to \mathbb{R}^n$, which is a non-orientable minimal surface. And conversely, all non-orientable minimal surfaces are expressed in the above way.

Lemma 3. *Let $\beta(s)$, $B(s)$ be Björling data such that $\beta(s)$ is T-periodic and $B(s)$ is T-antiperiodic, i.e. $B(s+T) = -B(s)$ for all s. Then the minimal surface generated by these data has in a neighbourhood of $\beta(s)$ the topology of a Möbius strip, with fundamental group spanned by $\beta(s)$.*

Conversely, every minimal Möbius strip in $\mathbb{R}^n$ is generated in this way.

Proof. Let $\psi' : \mathcal{M} \to \mathbb{R}^n$ be a minimal immersion of a Möbius strip $\mathcal{M}$, and choose a regular analytic closed curve $\tilde{\Gamma} \subset \mathcal{M}$ that spans the fundamental group of $\mathcal{M}$.

Let Σ be the two sheeted cover of $\mathcal{M}$ endowed with its canonical Riemann surface structure, and let $I : \Sigma \to \Sigma$ be its associated antiholomorphic involution without fixed points.

In this way, there exists a regular analytic closed curve $\Gamma \subset \Sigma$ spanning the fundamental group of Σ, given by $p(\Gamma) = \tilde{\Gamma}$. Once here, we can parametrize Γ as $\Gamma(s) : \mathbb{R} \to \Gamma$, where $\Gamma(s)$ is $2T$-periodic for some $T > 0$, and verifies $I(\Gamma(s)) = \Gamma(s+T)$ for all s. Particularly, $\tilde{\Gamma}(s) = p(\Gamma(s))$ is T-periodic.

Consider now $\psi : \Sigma \to \mathbb{R}^n$ given by $\psi = \psi' \circ p$. Then $\beta(s) = \psi(\Gamma(s))$ is T-periodic. Besides, if $\Pi(s)$ denotes the distribution of oriented tangent planes of $\psi : \Sigma \to \mathbb{R}^n$ along $\beta(s)$, it happens that $\Pi(s+T)$ agrees with $\Pi(s)$, but with opposite orientation. Therefore, if $B(s) = J\beta'(s)$ is the $\pi/2$-rotation of $\beta'(s)$ in the oriented plane $\Pi(s)$, we obtain $B(s+T) = -B(s)$ for all $s \in \mathbb{R}$.

Conversely, take $\beta(s)$, $B(s)$ Björling data so that $\beta(s)$ is T-periodic and $B(s)$ is T-antiperiodic. In this case, arguing as in [6,7], the solution to this Björling problem, given by (2.1), is defined on the quotient $\Omega/(2T\mathbb{Z})$, being Ω the open set of $\mathbb{C}$ determined by Theorem 2. Clearly, Ω contains $\mathbb{R}$, and we can assume that it verifies the symmetry condition $\Omega = \Omega^*$, being $\Omega^* = \{\bar{z} : z \in \Omega\}$. Now observe that on $\Omega/(2T\mathbb{Z})$ we can define the antiholomorphic involution $I : \Omega/(2T\mathbb{Z}) \to \Omega/(2T\mathbb{Z})$ without fixed points given by

$$I(z) = \bar{z} + T. \tag{2.2}$$

In addition, the map $\psi : (\Omega/(2T\mathbb{Z}), I) \to \mathbb{R}^n$ given by (2.1) verifies $\psi = \psi \circ I$, and thus defines a minimal Möbius strip. □

3. Trigonometric Björling data

From this point until the end of the paper we investigate complete minimal surfaces of finite total curvature in $\mathbb{R}^n$ by means of the solution to Björling problem.

Let $\psi : \Sigma \to \mathbb{R}^n$ be a complete minimal surface that has *finite total curvature*, that is, its total curvature $\mathcal{C}(\Sigma) = \int_\Sigma K \, \mathrm{d}A$ has finite (negative) value. Here $\mathrm{d}A$ is the area element associated to the metric $\mathrm{d}s^2$ of the surface Σ. Then Σ has the conformal type of a compact Riemann surface minus a finite number of points, and the Weierstrass one-form $\Phi = \psi_z \, \mathrm{d}z$ extends meromorphically to those points, called *ends* of the surface. In addition, we have the quantification $\mathcal{C}(\Sigma) = -2\pi m$, for $m \in \mathbb{N}$. More exactly, if $\Sigma \cong \bar{\Sigma} \setminus \{p_1, \dots, p_r\}$ where $\bar{\Sigma}$ is a compact Riemann surface of genus g, and μ_j denotes the order of the singularity of Φ at p_j, then

$$\mathcal{C}(\Sigma) = -2\pi \left(2g - 2 + \sum_{j=1}^{r} \mu_j \right).$$

This expression is known as the *Jorge–Meeks formula*. We refer the reader to [13,8,14,9,12] for details regarding complete minimal surfaces of finite total curvature.

Apart from this formula, there is also an important inequality in the theory, due to Gackstatter [4]. Denote by $\mathrm{Dim}(\psi)$ the dimension of the smallest affine subspace of $\mathbb{R}^n$ that contains the image of $\psi : \Sigma \to \mathbb{R}^n$. Then *Gackstatter's inequality* asserts that

$$\mathrm{Dim}(\psi) \leq 3 - r - 4g - \frac{\mathcal{C}(\Sigma)}{\pi}. \tag{3.1}$$

The case of complete non-orientable minimal surfaces of finite total curvature is studied by passing to the two-sheeted oriented covering and applying there the above theory, keeping control of the antiholomorphic involution appearing in the non-orientable description. In particular, in this context there are analogous relations to the Jorge–Meeks formula and Gackstatter's inequality. We refer to [12] for the details.

Next, we turn back to the Björling problem, proposing the following definition.

Definition 4. A pair of Björling data $\beta(s)$, $B(s)$ in $\mathbb{R}^n$ is said to be *trigonometric* if there exist $\alpha_m, \sigma_m, \delta_m, \gamma_m$ in $\mathbb{R}^n$, $1 \leq m \leq N$ such that

$$\beta'(s) = \sum_{m=1}^{N} \{\alpha_m \cos(ms) + \sigma_m \sin(ms)\}, \tag{3.2}$$

and

$$B(s) = \sum_{m=1}^{N} \{\delta_m \cos(ms) + \gamma_m \sin(ms)\} + \delta_0. \tag{3.3}$$

The trigonometric data $\beta(s)$, $B(s)$ are *non-null* if their holomorphic extensions on $\mathbb{C}$ verify $\beta'(z) - \mathrm{i}B(z) \neq 0$ for all z.

By means of this concept we show next a geometric description of the complete minimal cylinders of finite total curvature in $\mathbb{R}^n$.

Proposition 5. *Let $\beta(s)$, $B(s)$ be non-null trigonometric data in $\mathbb{R}^n$ given by* (3.2) *and* (3.3), *and assume that the condition*:

$$(\alpha_m, \sigma_m) \neq \pm(\gamma_m, -\delta_m) \tag{3.4}$$

is fulfilled for some $m \geq 1$. Then the solution to Björling problem for the data $\beta(s)$, $B(s)$ given by Theorem 2 *is a complete minimal surface $\psi : \mathbb{C}/(2\pi\mathbb{Z}) \to \mathbb{R}^n$ with the topology of a cylinder and finite total curvature $-2\pi(N + N')$, being*

$$N = \max\{m \in \mathbb{N} : (\alpha_m, \sigma_m) \neq (\gamma_m, -\delta_m)\},$$

$$N' = \max\{m \in \mathbb{N} : (\alpha_m, \sigma_m) \neq -(\gamma_m, -\delta_m)\}.$$

Conversely, every complete minimal cylinder of finite total curvature in $\mathbb{R}^n$ can be constructed following this process.

Proof. Let $\psi : \Sigma \to \mathbb{R}^n$ be a complete minimal surface of finite total curvature with the topology of a cylinder. Then, as Σ is conformally equivalent to a finitely punctured compact Riemann surface, we must have $\Sigma \equiv \mathbb{C} \setminus \{0\}$. In addition, its Weierstrass one-form extends meromorphically to $\mathbb{C} \cup \{\infty\}$. Hence, if w denotes the complex parameter of $\mathbb{C} \setminus \{0\}$, it holds $\Phi = (\tilde{\phi}_1(w), \dots, \tilde{\phi}_n(w))\, \mathrm{d}w$, where each $\tilde{\phi}_j(w)$ is of the form:

$$\tilde{\phi}_j(w) = \frac{p_j(w)}{w^{k_j}} \tag{3.5}$$

being $p_j(w)$ a polynomial with $p_j(0) \neq 0$, and $k_j \in \mathbb{N}$.

As Φ has no real periods, completeness of ψ at the end 0 is equivalent to the condition $k_j \geq 2$ for some $j \in \{1, \dots, n\}$. In addition, the change of variable $\zeta = 1/w$ shows that completeness at ∞ is equivalent to degree$(p_j(w)) \geq k_j$ for some $j \in \{1, \dots, n\}$.

Besides, the Jorge–Meeks formula for minimal surfaces in $\mathbb{R}^n$ provides:

$$\mathcal{C}(\Sigma) = -2\pi(-2 + \mu_1 + \mu_2), \tag{3.6}$$

where μ_1 (resp. μ_2) is the order of Φ at the end 0 (resp. ∞). Now, by (3.5) we obtain directly that

$$\mu_1 = \max\{k_j\}_{1 \leq j \leq n} \tag{3.7}$$

and

$$\mu_2 = \max\{\text{degree}(p_j) + 2 - k_j\}_{1 \leq j \leq n}. \tag{3.8}$$

To relate these facts with the Björling problem, we identify $\mathbb{C} \setminus \{0\}$ with the quotient $\mathbb{C}/(2\pi\mathbb{Z})$ by means of

$$w(z) = \mathrm{e}^{\mathrm{i}z}. \tag{3.9}$$

In this way, our surface is expressed as $\psi : \mathbb{C}/(2\pi\mathbb{Z}) \to \mathbb{R}^n$. As Φ extends meromorphically to the compactification of $\mathbb{C}/(2\pi\mathbb{Z})$, it must be of the form $\Phi = \phi(z)\, \mathrm{d}z$ for

$$\phi(z) = \sum_{-N'}^{N} \mathbf{c}_m\, \mathrm{e}^{\mathrm{i}mz}, \tag{3.10}$$

being $\mathbf{c}_m \in \mathbb{C}^n$ for each $m \in \{-N', \dots, N\}$. In other words, $\phi(z)$ is a vector trigonometric polynomial. Furthermore, we shall assume without loss of generality that $\mathbf{c}_N \neq 0$, and $\mathbf{c}_{-N'} \neq 0$, and observe that $\phi(z)$ may be written alternatively as

$$\phi(z) = \mathbf{c}_0 + \sum_{m=1}^{\max\{N, N'\}} \{\mathbf{a}_m \cos(mz) + \mathbf{b}_m \sin(mz)\}, \tag{3.11}$$

where $\mathbf{a}_m = \mathbf{c}_m + \mathbf{c}_{-m}$ and $\mathbf{b}_m = \mathrm{i}(\mathbf{c}_m - \mathbf{c}_{-m})$.

In addition, from the above computations the completeness of the surface is equivalent to the condition:

$$\mathbf{c}_m \neq 0 \quad \text{for some}\, m \geq 1 \quad \text{and} \quad \mathbf{c}_{-m'} \neq 0 \quad \text{for some}\, m' \geq 1. \tag{3.12}$$

Moreover, the total curvature of ψ takes the value $\mathcal{C}(\Sigma) = -2\pi(N + N')$.

Now denote $\beta(s) = \psi(s, 0)$ and $B(s) = (\partial\psi/\partial t)(s, 0)$, where $z = s + it$. Then $\phi(z) = 2(\partial\psi/\partial z)$, and so by (3.10) we see that $\beta(s)$, $B(s)$ are trigonometric Björling data. Moreover, the regularity of the surface indicates that these data are non-null. Specifically, following the notation in (3.10), we obtain that $\beta'(s)$ and $B(s)$ are given, respectively, by (3.2) and (3.3) for

$$\alpha_m - \mathrm{i}\delta_m = \mathbf{c}_m + \mathbf{c}_{-m}, \qquad \sigma_m - \mathrm{i}\gamma_m = \mathrm{i}(\mathbf{c}_m - \mathbf{c}_{-m}), \qquad \delta_0 = \mathbf{c}_0.$$

From here, it is straightforward to check that completeness of the surface, defined by (3.12), is expressed alternatively as (3.4). Moreover, the total curvature is described as we specified above: $\mathcal{C}(\Sigma) = -2\pi(N + N')$.

Conversely, if $\beta(s)$, $B(s)$ are trigonometric Björling data, then we know that the surface they describe is parametrized as $\psi : \mathbb{C}/(2\pi\mathbb{Z}) \to \mathbb{R}^n$, and its Weierstrass one-form Φ extends meromorphically to the ends $\pm\infty$. Furthermore, as the data are non-null, the surface is regular, and by (3.4) it is complete. Finally, the computation of the value of the total curvature is direct. □

By means of this result we can explore the situation referred to complete minimal Möbius strips of finite total curvature.

Theorem 6. *Let $\beta(s)$, $B(s)$ be non-null trigonometric Björling data so that $\beta(s)$ only has terms of even order, and $B(s)$ only has terms of odd order. Then the solution to the Björling problem in $\mathbb{R}^n$ for the data $\beta(s)$, $B(s)$ described by Theorem 2 is a complete minimal Möbius strip with finite total curvature $-2\pi \max\{N_1, N_2\}$, being N_1 (resp. N_2) the degree of the trigonometric polynomial $\beta(s)$ (resp. $B(s)$).*

Conversely, all complete minimal Möbius strips in $\mathbb{R}^n$ of finite total curvature are recovered in this way.

Proof. Let $\psi' : \mathcal{M} \to \mathbb{R}^n$ be a minimal Möbius strip in the conditions of the theorem. Then $\mathcal{M}$ is homeomorphic to a once punctured projective plane $\mathbb{R}\mathbf{P}^2 \setminus \{q\}$, and the two-sheeted covering Σ of this surface has the topology of a cylinder. As this covering must have the conformal type of a twice punctured Riemann sphere, we may assume that $\Sigma \equiv \mathbb{C} \setminus \{0\}$, and hence its *double surface* is $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{R}^n$. In addition, the antiholomorphic involution without fixed points under which $\mathcal{M}$ is recovered is given by $\tilde{I}(w) = -1/\bar{w}$.

Moreover, if we make the change of parameter (3.9), the surface is expressed as $\psi : (\mathbb{C}/(2\pi\mathbb{Z}), I) \to \mathbb{R}^n$, where $I : \mathbb{C}/(2\pi\mathbb{Z}) \to \mathbb{C}/(2\pi\mathbb{Z})$ is defined as (2.2).

Since the surface is complete of finite total curvature, by Theorem 5, if we define $\beta(s) = \psi(s, 0)$ and $B(s) = (\partial\psi/\partial t)(s, 0)$, where $z = s + it$, then $\beta(s)$, $B(s)$ are non-null trigonometric Björling data.

But now we get from Lemma 3 that $\beta(s)$ is π-periodic, and $B(s)$ is π-antiperiodic. This ensures that $\beta(s)$, given by (3.2), only has terms of even order, i.e. $\alpha_m = \sigma_m = 0$ for every odd number m. On the other hand $B(s)$, given by (3.3), has all its terms of odd order.

Since $\beta(s)$ is not constant, the completeness condition (3.4) is fulfilled. The value of the total curvature follows from Proposition 5 and the obvious relation $\mathcal{C}(\mathcal{M}) = \mathcal{C}(\Sigma)/2$.

The converse is easily obtained from Lemma 3 and Proposition 5. □

4. Möbius strips of critical total curvature

Let us begin this last section with the construction of some basic complete minimal Möbius strips by means of the solution to adequate Björling problems.

Example 7 (Meeks minimal Möbius strip). The easiest way to take Björling data which generate a Möbius strip by means of Lemma 3 is to choose a circle as curve, and suppose that the tangent

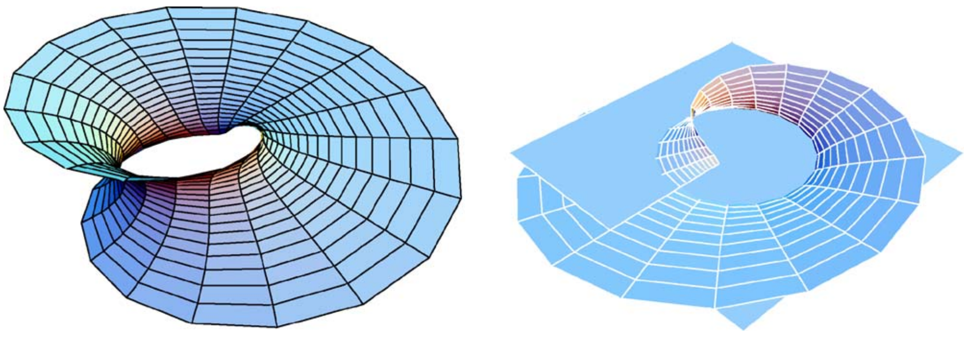

Fig. 1. Meeks minimal Möbius strip in $\mathbb{R}^3$, and its generating circle.

plane of the surface along this circle turns at a constant angular speed, so that it describes a complete turn for every two turns of the circle. In other words, let us take the data

$$\beta(s) = \frac{1}{2}(\cos(2s), \sin(2s), 0), \qquad B(s) = \cos s\,(\cos(2s), \sin(2s), 0) + \sin s\,(0, 0, 1).$$

Then, Theorem 6 indicates that the solution to this Björling problem is a minimal Möbius strip in $\mathbb{R}^3$ that is complete, regular, and has finite total curvature -6π. It was first obtained by Meeks [10] Fig. 1.

Example 8 (Oliveira's examples in $\mathbb{R}^3$). From a geometric viewpoint, it is easy to generalize the Björling data of Meeks minimal Möbius strip so that the new data still generate a minimal Möbius strip in $\mathbb{R}^3$. For this, we ask the circle to trace an even number of turns for each turn of the tangent plane. Analytically, this means to consider the Björling data:

$$\beta(s) = \frac{1}{2k}(\cos(2ks), \sin(2ks), 0), \qquad B(s) = \cos s\,(\cos(2ks), \sin(2ks), 0) + \sin s\,(0, 0, 1).$$

Again, the solution to this Björling problem is a minimal Möbius strip in $\mathbb{R}^3$ that is regular and complete. By Theorem 6 its total curvature is $-2\pi(2k+1)$. These examples were obtained by Oliveira in [12].

Example 9 (Möbius strips in $\mathbb{R}^4$). Motivated by Meeks and Oliveira's examples, we turn our attention to $\mathbb{R}^4$ and define the data:

$$\beta(s) = \frac{1}{2k}(\cos(2ks), \sin(2ks), 0, 0), \qquad B(s) = (0, 0, \cos s, \sin s).$$

These Björling data generate a complete, regular minimal Möbius strip that lies fully in $\mathbb{R}^4$, and has finite total curvature $-4\pi k$. All these examples were obtained by Oliveira [12].

Our final result concerns complete minimal Möbius strips that have finite total curvature of critical value for the non-orientable Gackstatter inequality. This inequality was obtained in [12] as a non-orientable analogue of Gackstatter's inequality (3.1), and for the case of Möbius strips (i.e. once punctured projective planes) reduces to

$$\mathrm{Dim}(\psi) \leq -\frac{\mathcal{C}(\mathcal{M})}{\pi}. \tag{4.1}$$

Definition 10. A complete minimal Möbius strip in $\mathbb{R}^n$ has *critical total curvature* if it attains equality in Gackstatter's inequality (4.1).

In particular, a Möbius strip of critical total curvature always verifies that $\mathrm{Dim}(\psi)$ is even. For instance, the minimal Möbius strip in Example 9 has critical total curvature -4π.

We intend to determine all the complete minimal Möbius strip that have critical total curvature. For this, the following lemma is quite useful.

Lemma 11. *Let* $\mathbf{f}(s) = \mathbf{a} + \sum_{k=1}^{m} \mathbf{g_k}\cos(ks) + \mathbf{h_k}\sin(ks)$ *be a vector trigonometric polynomial of degree m in* $\mathbb{R}^n$*, and let N be the rank of the family in* $\mathbb{R}^n$:

$$\mathcal{F} = \{\mathbf{g_1}, \mathbf{h_1}, \ldots, \mathbf{g_m}, \mathbf{h_m}\}. \tag{4.2}$$

Then $\mathrm{Dim}(\mathbf{f}(s)) = N$*, being* $\mathrm{Dim}(\mathbf{f}(s))$ *the dimension of the smallest affine subspace of* $\mathbb{R}^n$ *that contains the image of* $\mathbf{f}(s)$.

Proof. Obviously, the image of $\mathbf{f}(s) - \mathbf{a}$ lies in the subspace spanned by $\mathcal{F}$. Hence, $\mathrm{Dim}(\mathbf{f}(s)) \leq N$. But now, if $\mathrm{Dim}(\mathbf{f}(s)) < N$, there would exist a vector $\mathbf{v}$ in the subspace spanned by $\mathcal{F}$ so that $\langle \mathbf{f}(s), \mathbf{v}\rangle$ is constant. This would imply that $\mathbf{v}$ is orthogonal to $\mathcal{F}$, which is impossible. □

Theorem 12. *Let* $\beta(s) : \mathbb{R} \to \mathbb{R}^{2n}$ *and* $B(s) : \mathbb{R} \to \mathbb{R}^{2m}$ *be vector trigonometric polynomials, where* $m = n$ *if n is even and* $m = n + 1$ *if n is odd, such that*

1. $\beta(s)$ *has degree* $2n$*, only has terms of even order and the family of its coefficients is linearly independent.*
2. $B(s)$ *has degree* $2m - 1$*, only has terms of odd order and the family of its coefficients is linearly independent.*
3. $\langle \beta'(s), \beta'(s)\rangle = \langle B(s), B(s)\rangle > 0$*, and the entire extensions* $\beta(z)$*,* $B(z)$ *of* β*,* B *do not vanish simultaneously on* $\mathbb{C}$.

Then β*,* B *are Björling data in* $\mathbb{R}^{2n} \times \mathbb{R}^{2m} \equiv \mathbb{R}^{2(n+m)}$*, and the minimal surface they generate is a complete minimal Möbius strip that has critical total curvature of value* $-2\pi(m + n)$.

Conversely, every complete minimal Möbius strip of critical total curvature is constructed in this way.

Proof. Let $\psi : (\mathbb{C}/(2\pi\mathbb{Z}), I) \to \mathbb{R}^l$ be a complete minimal immersion of the Möbius strip $\mathcal{M} = (\mathbb{C}/(2\pi\mathbb{Z}), I)$ in some $\mathbb{R}^l$, and assume that it has critical total curvature of value $-2\pi k$. Thus, in particular, ψ lies *fully* in some $\mathbb{R}^{2k} \subset \mathbb{R}^l$, i.e. it does not lie in any proper affine subspace of this $\mathbb{R}^{2k}$. Hence, Theorem 6 guaranties that ψ can be recovered as the solution to a Björling problem in $\mathbb{R}^{2k}$ with trigonometric initial data $\beta(s)$, $B(s)$, so that

1. $\beta(s)$ only has terms of even order, and $B(s)$ only has terms of odd order, and
2. the total curvature of ψ is $-2\pi \max\{N_1, N_2\}$, being N_1 (resp. N_2) the degree of $\beta'(s)$ (resp. $B(s)$).

Thus, $k = \max\{N_1, N_2\}$.

Suppose that $N_1 > N_2$, and so $k = N_1$. Hence, k is even, and as $\beta(s)$ only has terms of even order, by Lemma 11 we obtain $\mathrm{Dim}(\beta'(s)) \leq k$. In addition, as $N_2 \leq k - 1$ and $B(s)$ only has terms of odd order, it happens that $\mathrm{Dim}(B(s)) \leq k$. From this and since ψ lies fully in $\mathbb{R}^{2m}$, we get $\mathrm{Dim}(\beta'(s)) = \mathrm{Dim}(B(s)) = k$. So, by Lemma 11 the degree of $\beta'(s)$ is k, the degree of $B(s)$ is $k - 1$, and their families of coefficients are linearly independent.

Suppose now that $N_2 > N_1$. So $k = N_2$, an odd number. Acting as above we infer that $\mathrm{Dim}(B(s)) \leq k + 1$ and $\mathrm{Dim}(\beta'(s)) \leq k - 1$. Again, as the surface is full we obtain that

$\mathrm{Dim}(B(s)) = k + 1$, $\mathrm{Dim}(\beta'(s)) = k - 1$, $B(s)$ has degree k, $\beta'(s)$ has degree $k - 1$ and their families of coefficients are linearly independent.

Finally, as it is obvious that the subspaces spanned by $\beta'(s)$ and $B(s)$ generate $\mathbb{R}^{2k}$ and are orthogonal, we can identify $\mathrm{span}(\beta'(s)) \equiv \mathbb{R}^{2n}$, $\mathrm{span}(B(s)) \equiv \mathbb{R}^{2m}$, and $\mathbb{R}^{2k} \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2m}$, and the numbers n, m verify the conditions of the theorem.

The converse follows directly from Lemma 11 and Theorem 6. $\square$

In [12, Theorem 2.11], Oliveira found for each $m \geq 2$ an example of a complete minimal Möbius strip of critical total curvature lying fully in $\mathbb{R}^{2m}$. The following example generalizes those ones by means of Theorem 12.

Example 13 (Möbius strips of critical total curvature). Consider $m \geq 2$ and $\varepsilon \in \{0, 1\}$ so that $m + \varepsilon$ is even, and take non-null numbers $c_l, d_l \in \mathbb{R}$ verifying

$$4 \sum_{l=1}^{(m-\varepsilon)/2} l^2 c_l^2 = \sum_{l=1}^{(m+\varepsilon)/2} d_l^2 > 0.$$

Define additionally the curve $\beta(s) : \mathbb{R} \to \mathbb{R}^{m-\varepsilon}$ as

$$\beta(s) = (c_1 \cos(2s), c_1 \sin(2s), \ldots, c_{(m-\varepsilon)/2} \cos((m - \varepsilon)s), c_{(m-\varepsilon)/2} \sin((m - \varepsilon)s))$$

and the field $B(s) : \mathbb{R} \to \mathbb{R}^{m+\varepsilon}$ given by

$$B(s) = (d_1 \cos(s), d_1 \sin(s), \ldots, d_{(m+\varepsilon)/2} \cos((m + 1 - \varepsilon)s), d_{(m+\varepsilon)/2} \sin((m + 1 - \varepsilon)s)).$$

Then, if we identify $\mathbb{R}^{2m} \equiv \mathbb{R}^{m-\varepsilon} \times \mathbb{R}^{m+\varepsilon}$, it holds that $\beta(s)$, $B(s)$ are non-null trigonometric Björling data in $\mathbb{R}^{2m}$ that satisfy the conditions in Theorem 12. So, the solution to this Björling problem is a complete minimal Möbius strip in $\mathbb{R}^{2m}$ with critical total curvature of value $-2\pi m$.

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